



# MathPath 2020 Qualifying Test

## Instructions

*Do NOT start work on this test until you have read these instructions, and both you and a parent or guardian have also read the Certification Form, which appears on page three of this document. Later, just before you submit your work, you will have to sign the Certification Form in the presence of your parent or guardian.*

### Rules about getting help

The Certification Form summarizes the rules about getting help. For more details, which explain why we insist on these rules, see [www.mathpath.org/QTheprules.htm](http://www.mathpath.org/QTheprules.htm).

### Electronic calculations

Except where explicitly disallowed, you may use a calculator or computer to perform calculations you find tedious, but you must say so each time. For instance, if for some reason you decide you need to compute  $1.234/2.345$ , you can write

$$1.234/2.345 = .5262 \text{ (calculator).}$$

If you use a computer program to get an answer by brute force (trying every possibility), you won't get much credit, even if the answer is right.

### Rules for writing up your answers

- If you handwrite your work, write it on standard  $8.5 \times 11$  American letter paper or A4 international paper, ruled or unruled. If you use ruled paper, either use wide ruling or double space. Your writing must be *large, dark, legible*, and *have ample margins*. Do not use a pencil if you write lightly. Do not smudge up the paper with erasures. Use a pen with dark ink and cross out instead.
- All handwritten submissions will be scanned, by us if not by you. Therefore, if you submit hardcopy, do not staple, paperclip, bind in a folder, etc. We must be able to put your submission into our scanner immediately upon receipt.
- Use only one side of each sheet. Your name and the page number should appear on the top of each sheet. The problems should also be numbered. Please start each problem on a new sheet. Many of these problems have several parts, which can be answered on the same sheet. You need not copy the statements of problems.
- Generally we think that earlier problems are easier, but problems vary a lot in difficulty, as do parts of problems.
- You may write your documents electronically, say with L<sup>A</sup>T<sub>E</sub>X or the math facility within MS Word. However, don't submit a L<sup>A</sup>T<sub>E</sub>X file or a Word file itself; *submit a single PDF file*. Also, do not create an electronic document if this means you will skip on algebra or on figures. (It is tempting to skip because complete work is harder to create electronically—you have to know the software well.)

## How to write well

Your reader can't ask you questions, so cryptic solutions or brief outlines will not do. Your solutions should show all the steps in your reasoning and in your computations. For full credit, you must justify your answers, even if the problem does not say "prove." For instance, an answer of the form "I observed a pattern and following this pattern the answer is..." will typically get only 1/3 credit—for seeking and finding the right pattern, but not explaining why that pattern must hold. The only time you do not have to prove what you claim would be when a problem indicates that no proof is required.

If you have never written a proof before, you may wonder if there's a special style you are supposed to use and don't know. Don't worry! First, there isn't just one correct style of proof, and second, we don't assume you have written proofs before. You'll learn about good styles for writing proofs when you get to MathPath. For now, think of proof as writing for a skeptical friend. You have to convince your friend that the mathematical statement is true, and that your approach is valid.

You also want to make your written solutions appealing and easy to understand. Long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred, but not if they are shorter simply by leaving out reasons. So, if your first solution to a problem is long and complicated, take time to see if you can find a shorter, more direct solution, and submit only the shorter solution. Mathematicians say that short, direct solutions are *elegant*.

Some submitted solutions will be displayed and discussed at camp (with names removed) as examples of good and not-so-good mathematical writing.

## How to submit

We prefer you to submit by email attachment, even if you handwrite your answers. But to make an electronic copy of handwritten work, you need access to a scanner, or scanner app, to create a single PDF file. *Photos of your printed pages (JPG files) are not acceptable.* Your scan must meet several other requirements, for instance about filename and scan settings to control file size. For all the details about any electronic submission to us, go to [www.mathpath.org/QTformatRequire.htm](http://www.mathpath.org/QTformatRequire.htm). Regardless, *the first page of your submission must be the Certification Form*, signed shortly before you submit. Submissions without the Certification Form will not be considered.

## Where to submit

If you wish to mail hardcopies of your solutions, address the envelope to

MathPath  
c/o April Verser  
265 N Oakhurst Ave, Apt 24  
Aurora IL 60504 (USA)

To submit scanned work electronically, email it to [admissions.assistant@mathpath.org](mailto:admissions.assistant@mathpath.org). We usually acknowledge receipt of the Qualifying Test within five days. If you wish to confirm receipt, you may inquire at the same email address. Whether you submit by mail or email, *always keep a copy of your work*. On rare occasions, submissions are lost in the mail or blocked in email.



# MathPath 2020 Qualifying Test Certification Form

*To be read promptly, then reread and signed  
just before submitting your finished work.*

I, \_\_\_\_\_, applicant to MathPath 2020, certify that:  
print full name

1. This submission is entirely my own work. I have discussed my work on specific problems with no one except the Past Executive Director, who responds to email about the interpretation of the test questions, and anyone he gave permission for me to ask (whom I've listed below). No one but myself has reviewed or edited this submission, nor have I even shown it to anyone, prior to submission.
2. Except where explicitly permitted within a QT problem, I have not looked at any resources, including online resources, in an effort to find out background information about any of the problems on this test. Nor have I tried to get any other help online, except from the MathPath website itself. (It is possible that you will come across information relevant to this test in the course of your normal math activities. As long as you were not seeking out test help, this is OK, but it still should be reported under #3 below.)
3. If for any reason I come across information that helps me solve any of the problems, or if I had already seen a problem very similar to a QT problem, I have listed those problems below, and in my solution for each such problem I have reported what information I found or remembered. (For instance, perhaps you remembered the statement of a key theorem but not how to prove it; or perhaps you remembered the solution method but not the answer.)
4. I understand that it is plagiarism if I learn how to solve a problem from some source and then submit a solution along those lines without crediting the source. It makes no difference if I copy from that source word for word or use my own words; if the ideas come from another source, it is plagiarism if no credit is given.
5. I understand that if MathPath staff find evidence that I have been untruthful in this Certification, that is grounds for denying admission or sending me home with no refund if MathPath 2020 is already in session.

Problem numbers of exceptions in item 3: \_\_\_\_\_

Person(s) I discussed QT with by permission: \_\_\_\_\_

To confirm this Certification, after finishing my solutions I have signed my name, and my parent or guardian has printed/signed their name, and dated this document, as my witness.

Applicant signature: \_\_\_\_\_

Parent/guardian printed name: \_\_\_\_\_

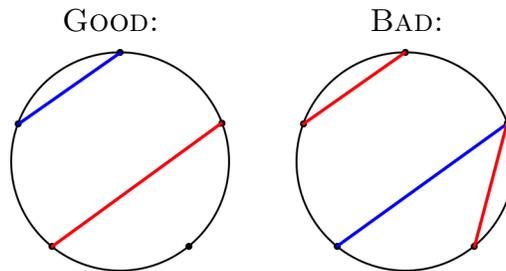
Signature: \_\_\_\_\_

Date: \_\_\_\_\_



# MathPath 2020 Qualifying Test Problems

1. You begin doodling on a diagram of five points spaced around a circle, using a red pen and a blue pen. How many ways are there to draw one or more segments, each of which is red or blue, so that none of the segments cross or share an endpoint? (The diagram on the right is bad because two of the segments share the same endpoint.)



It's fine to carefully list the possibilities, but it would be better to figure out how to count them all using just a few calculations and/or diagrams.

2. A certain type of edible wild mushroom is 99% water by weight, so after picking them mushroom lovers leave them out in the sun for a day to evaporate some of the water. Suppose that a collection of these mushroom, after such a drying, has been reduced to 98% water.

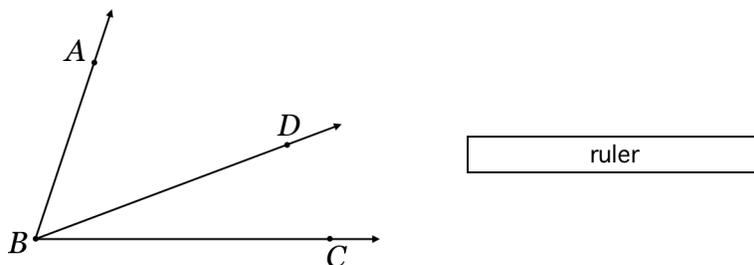
- (a) By what percent did the drying reduce the total weight of the mushrooms?
- (b) By what percent did the drying reduce the amount of water in the mushrooms?

3. A sequence of numbers  $a_1, a_2, a_3, a_4, \dots$  is defined by

$$\begin{aligned} a_1 &= 1 \cdot 1 \\ a_2 &= 1 \cdot 2 + 2 \cdot 1 \\ a_3 &= 1 \cdot 3 + 2 \cdot 2 + 4 \cdot 1 \\ a_4 &= 1 \cdot 4 + 2 \cdot 3 + 4 \cdot 2 + 8 \cdot 1 \\ &\vdots \end{aligned}$$

- (a) Use the pattern to write out an expression for  $a_5$ , then compute its value.
- (b) There is a short formula for  $a_n$  not involving a sum. Use whatever mathematical techniques you know to find or guess the formula.
- (c) If you are able to, prove that your formula works for all  $a_n$ .

4. We are given an acute angle  $\angle ABC$  with ray  $BD$  in its interior. Using only a “blank ruler” (a thin, unmarked rectangle), the goal is to construct points  $P$  on side  $BA$  and  $Q$  on side  $BC$  such that ray  $BD$  intersects  $PQ$  at a point  $R$  with  $PR = 2(QR)$ .



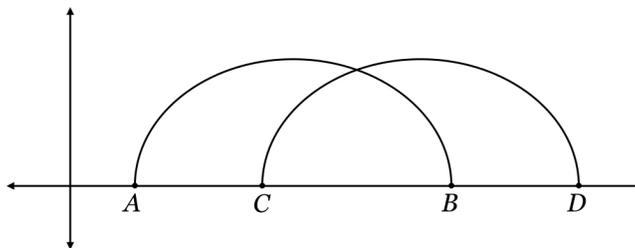
- Describe a method for constructing such points  $P$  and  $Q$ .
- Prove that your construction works. (Or at least indicate why it seems likely.)

5. Let us say that a function is **mathpathian** if it has a certain property we’ve discovered that’s hard to prove directly. However, after much work we have proved two important facts about mathpathian functions.

- If  $f(x)$  is mathpathian then so is  $-f(x)$ . For instance, if  $f_1(x) = x^2$  were mathpathian, then  $f_2(x) = -x^2$  would be too.
- If  $f(x)$  and  $g(x)$  are mathpathian, then so is  $\max(f(x), g(x))$ . (Recall that  $\max$  equals the larger of two values; or equals the common value if they are the same.)

- Prove that if  $f(x)$  is mathpathian, then so is  $|f(x)|$ . HINT: express  $|f(x)|$  using  $f$ ,  $-f$  and  $\max$ .
- Prove that if  $f(x)$  and  $g(x)$  are mathpathian, then so is  $\min(f(x), g(x))$ .

6. Let  $A, B, C$  and  $D$  be the points on the  $x$ -axis having coordinates  $A(1,0)$ ,  $B(6,0)$ ,  $C(3,0)$ , and  $D(8,0)$ . Draw a continuous curve from  $A$  to  $B$ , which means an unbroken path that can be drawn without lifting pencil from paper, as shown below. Next draw a copy of this path, obtained by shifting the first path two units to the right, which gives a path from  $C$  to  $D$ , also shown below. Either prove that these two paths must always intersect, or else find an example in which the paths do not intersect.



Suppose we change the distance between  $A$  and  $B$  (it was 5) and the distance between  $A$  and  $C$  (it was 2). Does the answer to the previous question—must the two paths intersect—ever change? Explore, conjecture, and give evidence for your conjecture.



## MathPath 2020 Qualifying Test Essay Questions

*Please answer each question below in one paragraph. Your responses will not affect your score on this Qualifying Test, but they are an important part of your application.*

- E1** Which problem on this Qualifying Test did you like the most, and why?
- E2** Which problem did you like the least, and why?
- E3** Of your work on this qualifying test, what are you most proud of, and why?
- E4** What is your greatest concern (worst fear) about attending MathPath? Explain.

In response to questions we receive about this test, we occasionally make clarifications. Questions seeking clarifications may be sent to [pastexecdir@mathpath.org](mailto:pastexecdir@mathpath.org). All clarifications to date can be found at [www.mathpath.org/clarify.htm](http://www.mathpath.org/clarify.htm).