MathPath 2019 Qualifying Test

Instructions

Do NOT start work on this test until you have read these instructions and you and a parent or guardian have read the Certification Form. The Certification Form is the third page of this packet. Later, just before you submit your work, you will have to sign this form in the presence of your parent or guardian.

Rules about getting help

Please read the five numbered items on the Certification Form. Those are the rules about getting help, short version. if you want a long version, which explains why we insist on these rules, go to www.mathpath.org/QThelprules.htm.

Electronic calculations

Except where explicitly disallowed, you are allowed to use a calculator or computer to do calculations you find tedious, but you must say so each time. For instance, if for some reason you decide you need to compute 1.234/2.345, you can write

$$1.234/2.345 = .5262 \text{ (calculator)}.$$

If you use a computer program to get an answer by brute force (trying every possibility), you won't get much credit, even if the answer is right.

Rules for writing up your answers

If you handwrite your work, write it on standard American letter paper (8.5×11) or A4 international paper, ruled or unruled. If you use ruled paper, either use wide ruling or double space. Your work must display well when projected on a screen, so the writing must be *large, dark, legible*, and *have ample margins*. Do not use a pencil if you write lightly. Do not smudge up the paper with erasures. Use a pen with dark ink and cross out instead.

All handwritten submissions will be scanned, by us if not by you. Therefore, if you submit hardcopy, do *not* staple, paperclip, bind in a folder, etc. We must be able to put your submission into our scanner immediately upon receipt.

Use only one side of each sheet. Your name and the page number should appear on the top of each sheet. The problems should also be numbered. Please start each problem on a new sheet. Most of these problems have parts – Part a), Part b), Part c) – and parts can be answered on the same sheet. You need not copy the statements of problems.

Generally we think that earlier problems are easier, but problems vary a lot in difficulty, as do parts of problems.

If you know how, you may write your documents electronically, say with LaTeX or the math facility within MS Word – but don't submit a LaTeX file or a Word file itself; *submit a single pdf file*. Do not create an electronic document if this means you will skimp on algebra or on figures. It is tempting to skimp because complete work is harder to create electronically – you have to know the software well.

How to write well

Communication is an important part of MathPath. Your reader can't ask you questions, so cryptic solutions or brief outlines will not do. Your solutions should show all the steps in your reasoning and in your computations. For full credit, you must justify (prove) your answers, even if the problem does not say prove. For instance, an answer of the form "I observed a pattern and following this pattern the answer is ..." will typically get only 1/3 credit – for looking for and seeing the right pattern but not explaining why that pattern must hold. The only time you do not have to prove what you claim is if a problem part says "No proof required" or the likes.

Many of you young students have never been asked to write proofs, and you may worry there's a special style you have to use and don't know. Stop worrying! First, there isn't just one correct style of proof, and second, we don't assume you have already written proofs. You'll learn about good styles for writing proofs when you get to MathPath. Actually this year we are starting to teach you right on the QT – see Problem 2. Other than that, for now think of proof as meaning what we already said: Justify the reasoning for all your steps as best you can. Think of yourself as writing for a skeptical friend. You have to convince your friend that your solution is correct.

You also want to make your written solutions appealing and easy to understand. Long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred, but not if they are shorter simply by leaving out reasons. So, if your first solution to a problem is long and complicated, take time to see if you can find a shorter direct solution, and submit the shorter solution only. Mathematicians say that short direct solutions are *elegant*.

Some submitted solutions will be displayed and discussed at camp (with names removed) as examples of good and not-so-good mathematical writing.

How to submit

We prefer you to submit by email attachment, even if you handwrite your answers. However, to make an electronic copy of handwritten work, you need access to a scanner to create a single pdf. *Photos of your printed pages (jpg files) are not acceptable.* Your scan must meet several other requirements, for instance about filename and scan settings to control file size. For all the details about any electronic submission to us, go to www.mathpath.org/QTformatRequire.htm.

Whether you submit electronically or hardcopy *the first page of your submission must be the Certification Form, signed shortly before you submit.* Submissions without the Certification will be rejected.

Where to submit

Check our website for updates, but for now send your hardcopy solutions to

MathPath c/o Prof Maurer
206 Benjamin West Ave
Swarthmore PA 19081-1421
(USA)

To submit your scanned work electronically, email it to mathpath.academics@gmail.com. We usually acknowledge receipt of the QT within 2 days of receipt. What if you do not receive an acknowledgment when you expect it? If you submitted electronically, please inquire with our Administrative Assistant at mathpath.academics@gmail.com. If you submitted hardcopy, please inquire with Prof Maurer at smaurer1@swarthmore.edu.

Whatever way you submit your work, *always keep a copy*. On rare occasions, submissions are lost in the mail or blocked in email.

2019 QT Certification

To be read promptly, then reread and signed just before submitting your finished work.

I, ____________________________________, applicant to MathPath 2019, certify that:
print name

1. This submission is entirely my own work. I have discussed my work on specific problems with no one except the MathPath Executive Director, who responds to email about the interpretation of the test questions, and anyone he gave permission for me to ask (whom I've listed below). No one but I has reviewed or edited this submission, nor have I even shown it to anyone, prior to submission.
2. Except where explicitly permitted within a QT problem, I have not looked up any materials, including online materials, in an effort to find out background information about any of the problems on this test or any background information about MathPath tests in general. Nor have I tried to get any other help online, except from the MathPath website itself. (It is just possible that you will come across information relevant to this test in the course of your normal math activities. As long as you were not seeking out test help, this is OK, but it still should be reported under #3 below.)
3. If for any reason I come across information that helps me solve any of the problems, or if I had already seen something like one or more of the problems, I have listed those problems below, and in my solution for each such problem I have reported what information I found or remembered. (For instance, perhaps you remembered the statement of the key theorem but not how to prove it; or perhaps you remembered the solution method but not the answer.)
4. I understand the following: It is plagiarism if I learn how to solve a problem from some source and then submit a solution along those lines without crediting the source. It makes no difference if I copy from that source word for word or use entirely my own words; if the ideas come from that source it is plagiarism if no credit is given.
5. I understand that if MathPath staff find evidence that I have been untruthful in this Certification, that is grounds for denying admission or sending me home with no refund if MathPath 2019 is already in session.

Problem Numbers of exceptions in Item 3: ____

Person(s) I discussed QT with by permission (Item 1):

____

To confirm this Certification, after finishing my solutions I have signed my name, and my parent or guardian has printed and signed his/her name, and dated this document, as my witness.

Applicant signature: ____________________________________

Parent or guardian printed name: ____________________________________

signature: ____________________________________

date: _____________________________

MathPath 2019 Qualifying Test
Problems

1. You and your friend play a game where you alternate taking stones from a pile, initially of n stones. At each turn the player removing stones gets to take any whole positive number of stones equal to or less than half of the remaining stones. When the player removing stones can't remove any, they lose. You go first.

a) (warm up question) For which pile sizes do you immediately lose, because you can't take anything?

b) For which n can you guarantee a win for yourself? How? Why not for other n?

c) Same problem, but the number of stones you can take on each turn is further restricted to strictly less than half the number remaining.

2. MathPath wants you to do proofs, but many of you have not yet been introduced to proof writing. In this problem we will illustrate how to write the most basic sort of proof, *proof from the definitions*. For some of you this will be new and will show us how well you pick up proof ideas. For others it may be a review and may give you a chance to polish your style. Some of you may have seen parts of this problem before. That's ok. Just be sure to inform us according to item 3 on the QT Certification form.

Definition. For positive integers a, b, we say that a **divides** b if $b = ka$ for some positive integer k.

Example of a proof using this definition:

Theorem. If a divides both b and c, then a divides $b + c$.

Proof: We are told to assume a divides b and c. From the definition of divides, that means there exist positive integers m and n such that $b = ma$ and $c = na$. We are asked to show that a divides $b + c$. By definition that means we must show there is a positive integer r such that $b + c = ra$. Adding the two equations we got from the assumption, we get $b + c = ma + na$. Furthermore, $ma + na = (m + n)a$ by a general property of real number multiplication (the distributive law). Thus $b + c = (m + n)a$. Since $m + n$ is a positive integer, we may choose $r = m + n$. Thus we have shown that there is a positive integer r such that $b + c = ra$ and so we have shown from the definition that a divides $b + c$. ∎ (end of proof symbol)

Proofs from the definitions are especially useful at the start of developing a theory, before you have many theorems you can invoke to justify your steps. The key to writing a correct proof is to use the definitions exactly as they are stated.

a) Prove from the definition that if a divides b and b divides c, then a divides c.

b) State and prove some other property of divides that you can prove from the definition.

3. **a**) Multiply out

$$(x + \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x - \sqrt{2} - \sqrt{3})$$

to get standard polynomial form, $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

This looks yucky, so normally it would be fine to expand it using software, but this time do it by hand. By hand doesn't necessarily mean the standard way. In fact, the briefer and simpler your computation (that is, the more ingenious and insightful), but still complete, the more points you will get. Besides, doing it by hand you may notice some patterns you may want to think about.

b) Extension. Again by hand, expand

$$(x+1+\sqrt{2}+\sqrt{3})(x+1+\sqrt{2}-\sqrt{3})(x+1-\sqrt{2}+\sqrt{3})(x+1-\sqrt{2}-\sqrt{3}).$$

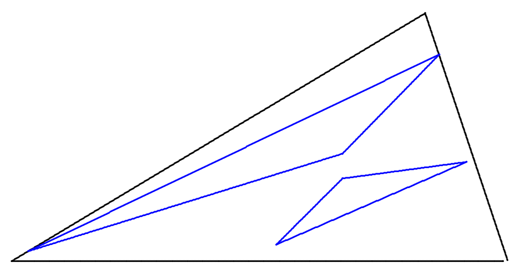
c) A number is called **algebraic** if it is a root of an equation $p(x) = 0$, where $p(x)$ is a polynomial with integer coefficients. For instance, $\phi = \frac{1+\sqrt{5}}{2}$ is algebraic because ϕ is a root of $x^2 - x - 1 = 0$.

In earlier parts of this problem, what numbers have you proved to be algebraic?

d) Generalize. That is, parts a) and b) are special cases of a more general result. Find, state, and justify as big a generalization of parts a) and b) as you can,

e) Conjecture. What other numbers do you think are algebraic? Any ideas how to prove it?

4. You are given a triangle, and can then draw any other triangle inside it, where "inside it" means the vertices of the second triangle are on or inside the first triangle. See figure.



Blue triangles inside a black triangle

It seems obvious that the perimeter of an inside triangle is less than the perimeter of the outside triangle. But George Thomas, the Founder of MathPath, has said, "A mathematician is cautious in the presence of the obvious." So is this inequality always true?

Prove or disprove it. If it's true, it should be provable using the following principle: the straight line segment between two points is the unique shortest path between them.

5. In this problem you answer only parts c), d) and e). Parts a) and b) are to be read in preparation for the rest.

a) There is a famous problem about picking socks out of a dresser in a dark room. You may already have heard some version of it. In one version, it goes like this:

You have a drawer full of individual socks, all the same size and style, but they come in 5 different colors. It is night and the power is out. You go to the dresser and start pulling out socks at random. How many do you have to pull out to guarantee you have at least one pair, where a pair is defined to be two socks of the same color? How many do you have to pull out to guarantee you have at least 2 pairs, where again, two socks must be the same color to be a pair, but the second pair can be a different color than the first?

b) A mathematician has a collection of quarters (American 25 cent coins) which she gives out randomly to 3 students. The students want raffle tickets more than quarters, but raffle tickets cost \$1, and the students have no opportunity to come together and share their money. Thus if one student has 7 quarters s/he can buy only one raffle ticket; 3 of the quarters are useless. Then, after they buy tickets, the students come together and count how many tickets they have bought. How many quarters does the mathematician have to give out so that, however she divides them up, the students will always be able to buy 5 raffle tickets among them?

c) Tell us whether or not you have seen Part a) and its solution before. How well do you remember the solution?

d) In some sense, problems a) and b) are the same. Explain as completely as you can. For instance, what in the raffle problem is the "same" as the colors in the sock problem?

e) Generalize. That is, state a problem that includes the problems of both a) and b) (without mentioning either socks or quarters) and solve it for any values of the numbers.

6. You wish to put the following knife's edge surface over the First Quadrant of the xy-plane. See the first two figures below. It should have height 0 over the positive x and y axes. Directly above the line $y = x$ on the xy-plane, it should rise at an angle of 45° from the origin. Call that rising line on the surface k (for knife's edge). Finally, between line k and the x-axis the surface is the plane containing these two lines, and between k and the y-axis it is the plane containing those two lines.

In principle the surface could extend to infinity, but to get helpful figures we have cut the surface off after a while and labeled various points.

Such a surface can be made by taking a piece of paper OABC and creasing it along the bisector of $\angle COA$, as shown in Figure 3. We need to know the value of angle α. It must be more than 45°. So the paper has an odd shape, one you would have to cut out from some standard piece of paper.

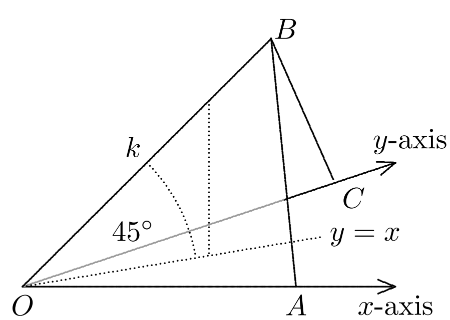


Fig 1. View from Quadrant IV

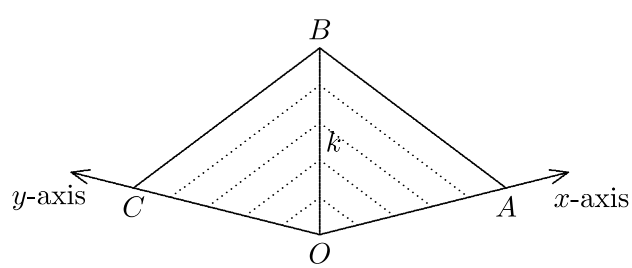


Fig 2. View from Quadrant III

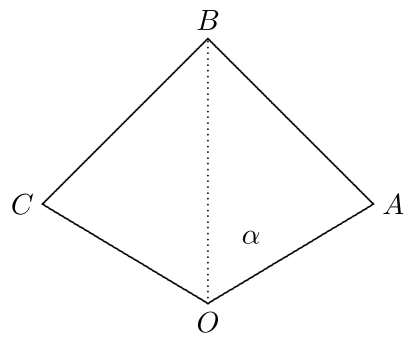


Fig. 3. Paper to be creased

a) What is the size of $\angle AOB$ in Fig. 1? (This is the size that one must make α in Fig. 3 in order for the paper to fit the surface when the paper is creased.)

b) Instead of creating a special piece of paper, it is possible to crease a square piece of origami paper (or even rectangular paper) at a corner and create a surface similar to what we have been discussing, but the knife's edge k can't rise so steeply and the paper's edges can't reach all the way to the axes. See Figs. 4–5. If the rise over the line $y = x$ is 30°, what is $\angle AOX$?

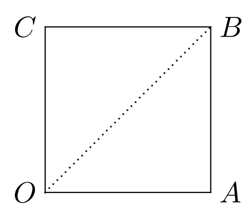


Fig. 4

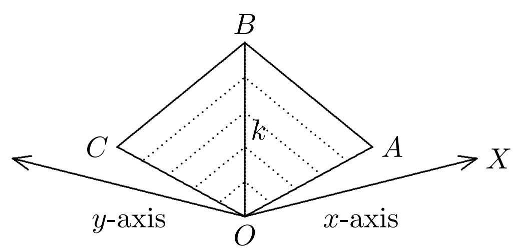


Fig.5. View from Quadrant III for Part b

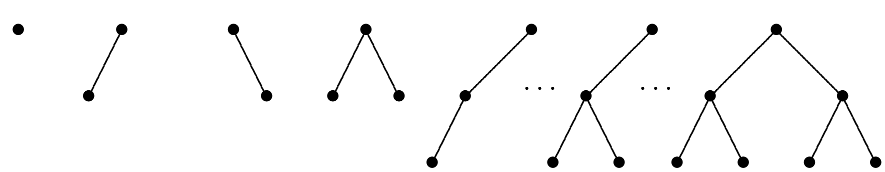
7. By a **graph** in this problem we mean a collection of nodes (also called vertices) and edges connecting them.

A **binary tree** is a special sort of graph which may be defined recursively as follows.

A single node is a binary tree.

A graph is a binary tree if it consists of a node, called the **root**; an edge hanging down left and/or an edge hanging down right; and at the other end of either such edge another smaller binary tree, attached at its root.

The picture shows the first several binary trees obtained using this definition.

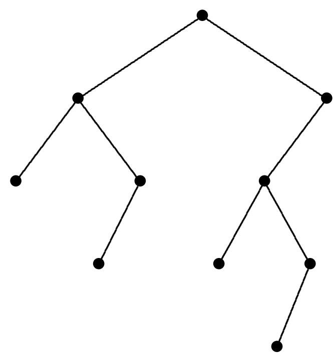


Alternatively, it turns out to be equivalent to define a binary tree as a connected graph where each node except one has exactly one edge coming in from above (the exception, the root, has no edges coming in from above) and each node has at most two edges going out below, a right edge and a left edge.

The **height** of a vertex in a binary tree is the length of the longest path going down from it, where the length of a path is the number of edges in it. The **left height** of a vertex (respectively, **right height**) is the length of the longest path down from a vertex starting with the left edge; if there is no left edge, the left height is 0. The height of a binary tree is the height of its root.

Finally, a binary tree is **nearly even** if at every vertex, The right and left heights differ by at most 1.

a) Consider the following binary tree. Determine the height and decide if the tree is nearly even.

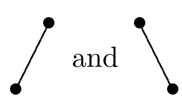


b) Determine with proof the maximum number of vertices $M(h)$ in a nearly even binary tree of height h.

c) Determine with proof the minimum number of vertices $m(h)$ in a nearly even binary tree of height h.

d) Prove or disprove: for every integer k such that $m(h) \leq k \leq M(h)$, there is a nearly even binary tree of height h and k nodes. *Warning*: you can't just randomly add edges one at a time starting with a minimal tree because you might cause the nearly even condition to be violated at some vertex.

e) How many different nearly even binary trees of height h are there with $m(h)$ vertices? Note: right and left are different in binary trees. So



are considered different binary trees because in one the root has a left edge and in the other a right edge.

Essay Questions

Please answer each question below in one or two paragraphs. Your answers will not affect your score on this QT, but they are another important part of your admissions folder.

E1. Which problem on this QT did you like the most and why? Which problem did you like the least and why?

E2. Why are you applying to MathPath? What do you hope to get out of MathPath that you are not already getting, or not getting enough of?

— end! —

Note: In response to questions we receive about this test, from time to time we make some clarification. All clarifications to date can be found at www.mathpath.org/clarify.htm.

Also, for examples of good and not-so-good solutions from earlier MathPath tests, go to www.mathpath.org/goodbadsols.htm.