MathPath 2017 Qualifying Test

Instructions

Do NOT start work on this test until you have read these instructions and you and a parent or guardian have read the Certification Form you will have to sign in the presence of your parent or guardian just before you submit your work. The Certification Form is the third page of this packet.

Rules about getting help

Please read the five numbered items on the Certification Form. Those are the rules about getting help, short version. if you want a long version, which explains why we insist on these rules, go to www.mathpath.org/QThelprules.htm.

Electronic calculations

You are allowed to use a calculator or computer to do calculations you find tedious, but you must say so each time. For instance, if for some reason you decide you need to compute 1.234/2.345, you can write

$$1.234/2.345 = .5262 \text{ (calculator)}.$$

The one exception to your having to tell us when you use a calculator is when the problem asks you to use one. This year, problems 4b and 6a ask you to use a calculator. Otherwise, everything can be done with paper and pencil.

If you use a computer program to get an answer by brute force (trying every possibility), you won't get much credit, even if the answer is right.

Rules for writing up your answers

If you handwrite your work, write it on standard American letter paper (8.5×11) or A4 international paper, ruled or unruled. If you use ruled paper, either use wide ruling or double space. Your work must display well projected on a screen, so the writing must be *large, dark, legible*, and *have ample margins*. Do not use a pencil if you write lightly. Do not smudge up the paper with erasures. Use a pen with dark ink and cross out instead.

All handwritten submissions will be scanned, by us if not by you. Therefore, if you submit hardcopy, do *not* staple, paperclip, bind in a folder, etc. We must be able to put your submission into our scanner immediately upon receipt.

Use only one side of each sheet. Your name and the page number should appear on the top of each sheet. The problems should also be numbered. Please start each problem on a new sheet. Most of these problems have parts — Part a), Part b), Part c) — and parts can be answered on the same sheet. You need not copy the statements of problems.

If you know how, you may write your documents electronically, say with LaTeX or the math facility within MS Word — but don't submit a LaTeX file or a Word file itself; *submit a single pdf file*. Do not create an electronic document if this means you will skimp on algebra or on figures. It is tempting to skimp because complete work is harder to create electronically — you have to know the software well.

How to write well

Communication is an important part of MathPath. Your reader can't ask you questions, so cryptic solutions or brief outlines will not do. Your solutions should show all the steps in your reasoning and in your computations. For full credit, you must justify (prove) your answers, even if the problem does not say prove. For instance, an answer of the form "I observed a pattern and following this pattern the answer is ..." will typically get only 1/3 credit – for looking for and seeing the right pattern but not explaining why that pattern must hold. The only time you do not have to prove what you claim is if a problem part says "No proof required" or the likes.

Many of you young students have never been asked to write proofs, and you may worry there's a special style you have to use and don't know. Stop worrying! First, there isn't just one correct style of proof, and second, we don't assume you have already written proofs. You'll learn about good styles for writing proofs when you get to MathPath. For now think of proof as meaning what we already said: Justify as best you can the reasoning for all your steps. Think of yourself as writing for a skeptical friend. You have to convince your friend that your solution is correct.

You also want to make your written solutions appealing and easy to understand. Long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred, but not if they are shorter simply by leaving out reasons. So, if your first solution to a problem is long and complicated, take time to see if you can find a shorter direct solution, and submit the shorter solution only. Mathematicians say that short direct solutions are *elegant*.

Some submitted solutions will be displayed and discussed at camp (with names removed) as examples of good and not-so-good mathematical writing.

How to submit

We prefer you to submit by email attachment, even if you handwrite your answers. However, to make an electronic copy of handwritten work, you need access to a scanner to create a single pdf. Photos of your printed pages (jpg files) are not acceptable. Your scan must meet several other requirements, for instance about filename and scan settings to control file size. For all the details about any electronic submission to us, go to www.mathpath.org/QTformatRequire.htm.

Whether you submit electronically or hardcopy *the first page of your submission must be the Certification Form, signed shortly before you submit.* Submissions without the Certification will be rejected.

Where to submit

Send your hardcopy solutions to

MathPath c/o Prof Maurer
206 Benjamin West Ave
Swarthmore PA 19081-1421
(USA)

To submit your scanned work electronically, email it to mathpath.academics@gmail.com.

We usually acknowledge receipt of the QT within 2 days of receipt. What if you do not receive an acknowledgment when you expect it? If you submitted electronically, please inquire with our Administrative Assistant at mathpath.academics@gmail.com. If you submitted hardcopy, please inquire with Prof Maurer at smaurer1@swarthmore.edu.

Whatever way you submit your work, *always keep a copy*. On rare occasions, submissions are lost in the mail or blocked in email.

2017 QT Certification

To be completed shortly before submission after you have finished your solutions

I, ______________________________, applicant to MathPath 2017, certify that:
print name

1. This submission is entirely my own work. I have discussed my work on specific problems with no one except the MathPath Executive Director and anyone he gave permission for me to ask (listed below). No one but I has reviewed or edited this submission, nor have I even shown it to anyone, prior to submission.
2. I have not looked up any materials, including online materials, in an effort to find out background information about any of the problems on this test or any background information about MathPath tests in general. Nor have I tried to get any other help online, except from the MathPath website itself.
3. If for any other reason I inadvertently came across information that helped me solve any of the problems, or if I had already seen something like one or more of the problems, I have listed those problems below, and in my solutions for each such problem I have detailed a) when and where I came across the information; b) what information I remembered. (For instance, perhaps you remembered the statement of the key theorem but not how to prove it; or perhaps you remembered the solution method but not the answer.)
4. I understand the following: It is plagiarism if I learn how to solve a problem from some source and then submit a solution along those lines without crediting the source. It makes no difference if I copy from that source word for word or use entirely my own words; if the ideas come from that source it is plagiarism if no credit is given.
5. I understand that if MathPath staff find evidence that I have been untruthful in this Certification, that is grounds for denying admission or sending me home with no refund if MathPath 2017 is already in session.

Problem Numbers of exceptions in Item 3: ______________________________

Person(s) I discussed QT with by permission (Item 1):

____

To confirm this Certification, after finishing my solutions I have signed my name, and my parent or guardian has printed and signed his/her name, and dated this document, as my witness.

Applicant signature: ______________________________

Parent or guardian printed name: ______________________________

signature: ______________________________

date: ______________________________

MathPath 2017 Qualifying Test
Problems

1. A famous trick problem goes:

> You travel from A to B at 40 mph (miles per hour). Then you travel from B to A (same route) at 60 mph. What is your average speed for the whole trip?

This is a trick problem because most people answer it right away but their answer is wrong!

a) Let us call that wrong answer the trick answer. What is the trick answer?

b) What is the right answer? Prove that you are right. Your proof should start by stating the definition of average speed for a whole trip.

c) If you have done everything correctly so far, your right answer is less than the trick answer. Was this chance? That is, with different speeds in the two halves, might the right answer be more than the trick answer? Or, is there a general principle here? Prove what you discover.

d) Change the original problem slightly so that the trick answer is now right! That is, fill in the dot-dot-dots below so that the trick answer is the correct answer, and prove that you are right.

> You travel . . . at 40 mph. Then you travel . . . at 60 mph. What is your average speed for the whole trip?

2. We introduce some notation for writing really big (but finite) numbers.

A **googol**, denoted g, is defined by $g = 10^{100}$. A **googolplex**, denoted G, is defined by $G = 10^g$. A **MathPatharoo**, denoted M, is defined by $M = G^G$.

Next, for positive integers m, n define $m \uparrow n$ (spoken "em uparrow en") as follows:

$$m \uparrow 1 = m$$
$$m \uparrow (n{+}1) = m^{m \uparrow n}.$$

For example, $4 \uparrow 2 = 4^{4\uparrow 1} = 4^4 = 256$.

a) Find the smallest integer n so that $2 \uparrow n \geq g$.

b) Find the smallest integer n so that $2 \uparrow n \geq M$.

As always, justify (prove) your answers, and try to keep your arguments clean and short; try to avoid really long or ugly calculations.

3. A positive integer n is **divisor rich** if the sum of its proper divisors is strictly greater than n. For instance, 10 is not divisor rich because its proper divisors are 1, 2, and 5, which add to 8, and $8 \not> 10$. Likewise 6 is not divisor rich because its proper divisors, 1, 2 and 3, add to exactly 6, not greater than 6. (A **proper divisor** of a positive integer n is a positive integer smaller than n that divides into n without remainder.)

a) Find the smallest divisor rich number. Show how you know it is smallest.

b) Prove: any integer multiple of a divisor rich number is divisor rich.

c) Is there any other class of numbers for which every integer multiple is divisor rich? (Look at your proof for Part b; maybe it accomplishes more than you have claimed so far.)

4. Every algebra student eventually learns the usual formula for the roots of the quadratic equation $ax^2 + bx + c = 0$, namely,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

Less well known is another formula for these same roots, a formula where the only square root is in the denominator.

a) Find this other formula. *Hint*: At some point most algebra students learn how to clear the radical from the denominator of fractions like $\frac{3}{4+\sqrt{5}}$. (If you don't know how to do this, read the discussion of problem 4 at www.mathpath.org/clarify.htm.) Now you need to clear the radical from the *numerator* of Eq. (1).

b) Using a standard scientific calculator (one that does calculations to a fixed number of digits you can't change) find the numerical value of the roots of $x^2 - 10^8x + 1 = 0$

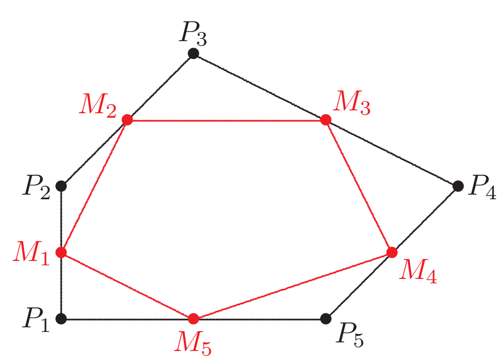
i) Using the standard formula,

ii) Using your new formula.

Something weird happens. What? Can you explain why it happens? (If nothing weird happens, your calculator may do calculations to more digits than most; replace the coefficient $b = 10^8$ in the quadratic you are solving with $10^9, 10^{10}$ and so on until something weird does happen.)

c) Welcome to the world of *numerical analysis* and *roundoff error*. Equivalent formulas may not be equivalent in practice. When might you want to use your new formula instead of the traditional formula?

5. For every polygon $P_1P_2 \cdots P_n$ there is a unique **midpoint polygon** defined as follows. For each side $P_1P_2, P_2P_3, \ldots, P_nP_1$ of the original polygon, draw in midpoint $M_1, M_2, \ldots, M_n$ and connect them in order $M_1M_2 \cdots M_nM_1$. The figure below shows an example where $n = 5$. The original polygon is black and the midpoint polygon is red.



Is this process reversible? Given an arbitrary polygon, is it always the midpoint polygon of some other polygon? Is that other polygon unique? Is there a simple construction to find it? Are there special cases of this question which are easier to answer? Show what you discover and what you prove.

6. Consider the positive integer powers of $\alpha = 1 + \sqrt{3}$, that is, α^n for $n = 1, 2, 3, \ldots$. Further, consider $\{\alpha^n\}$, the noninteger part of α^n, that is, what is left when you subtract off the greatest integer equal to or less than α^n. For instance, since $\alpha^3 \approx 20.392$, then $\{\alpha^3\} \approx .392$. Since α is irrational, one might expect the sequence $\{\alpha\}, \{\alpha^2\}, \{\alpha^3\}, \ldots$ to look random – any amount between 0 and 1 would be equally likely. So compute a number of terms of this sequence with a calculator.

a) What happens? Prove it.

b) Are there other irrational numbers β where the sequence $\{\beta\}, \{\beta^2\}, \{\beta^3\}, \ldots$ behaves similarly to $\{\alpha\}, \{\alpha^2\}, \{\alpha^3\}, \ldots$?

7. Your teacher makes you practice subtraction with a partner as follows. Your teacher picks a random positive integer, call it n. Then you pick a random integer s where $0 < s \leq n$. Your partner now has to subtract s from n, obtaining n_1. Then you pick another random integer s_1 where $0 < s_1 \leq n_1$ and again your partner has to do the subtraction, obtaining n_2. You keep repeating until the subtraction results in 0. This ends the round.

For instance, maybe your teacher gives you 23. You might pick 6 and your partner would subtract and obtain 17. Then you might pick 12 and your partner subtracts and gets 5. Then you might pick 4 and your partner gets 1. Then you have no choice but to pick 1; your partner subtracts and gets 0, ending the round. Your partner did four subtractions.

In the questions that follow, assume that you really pick your numbers at random. For example, when your teacher gave you 23, you picked any one of 1 to 23 with equal probability; and when your partner computed 17, you picked any one of 1 to 17 with equal probability. Also assume your partner never makes a subtraction mistake. The questions are about the average number of subtractions in a round, where each way a round can play out is counted according to how likely it is (i.e., the average is a weighted average).

a) For $n = 3$, there are 4 possible ways the first round of the practice session could run.

i) You pick 3 and the round ends after the first subtraction.

ii) You pick 2, your partner gets 1, and then you pick 1; there are two subtractions.

iii) There are two subtractions because you first pick 1 and then pick 2.

iv) There are three subtractions, because you pick 1 each time.

These rounds are not equally likely. Explain why scenario i) has probability 1/3 and scenario iv) has probability 1/6.

b) Show that the average number of subtractions when $n = 3$ is 11/6.

c) Determine the average number of subtractions when $n = 4$.

d) Find and prove a formula for the average number of subtractions for any integer $n \geq 1$.

e) Now instead of making you pick a random integer from 1 to the current number each time, your teacher makes it easier on you by giving you a spinner. The spinner picks integers equally likely from 1 to the original n every time, even when the current number is much smaller than n. If the spinner picks a number that would lead to a negative answer to the subtraction, the number is thrown out and you spin again.

For instance, suppose the teacher picks $n = 5$. The spinner will always choose an integer from 1 to 5. Suppose it picks 3. You partner subtracts and gets 2. Now the spinner might pick 4. You throw that result out since $4 > 2$. Now the spinner might pick 5. You throw that out too. Now it might pick 2. Your friend subtracts and gets 0, ending the round. There were two subtractions and four spins.

Question: As a function of the initial n, on average how many spins will there be in a round?

8. Briefly, which one of problems 1 to 7 did you like best and why? Which one of problems 1 to 7 did you like least and why?

— end! —

Note: In response to questions we receive about this test, from time to time we make some clarification. All clarifications to date can be found at www.mathpath.org/clarify.htm.

Also, for examples of good and not-so-good solutions from earlier MathPath tests, go to www.mathpath.org/goodbadsols.htm.