MathPath 2007 Qualifying Quiz

Instructions.

Do as many of the 6 problems below as you can. If you can do 3 problems (in whole or in sum of parts) you should definitely apply! Keep thinking about each problem until the approach to the solution comes to you, sometimes after several days. You may ask others to help you understand the statements of the problems, but the actual solutions must be your own. Do not be disappointed if you spend a long time on some problem but never solve it. This happens to famous mathematicians too!

Use 8.5×11 paper (ruled or unruled) and write on one side only. Please start each problem on a new sheet. The sheets and the problems should be numbered. You need not copy the statements of problems. However, your solutions should show all the steps in your reasoning and in your computations. The steps are more important than the answer. Correct answers without supporting reasoning will receive no credit.

Also, additional generalizations and additional solutions are welcome and may count significantly. That is, if you see one of these problems as a special case of a bigger problem, and can solve the bigger problem, do so. Or, if after solving the given problem completely, you see a completely different solution, submit it.

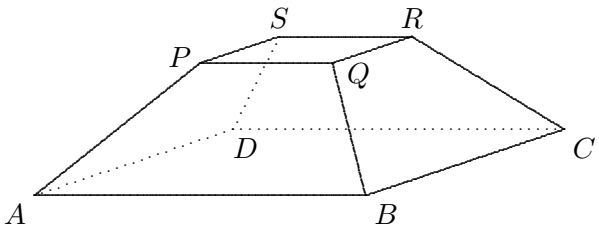
Communication is an important part of MathPath. Imagine you are writing to a friend who knows about as much mathematics as you but who has not thought about these problems before. You want your friend to understand your solutions as easily as possible, so your work must be clear. In particular, long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred (but not if they are shorter simply by leaving out reasons). So, if your first solution to a problem is long and complicated, see if you find a short direct solution, and submit only that. Mathematicians say that such short direct solutions are elegant.

For more discussion and examples of good and not-so-good solutions from earlier Math-Path quizzes, click HERE.

Click HERE for any posted hints and clarifications.

1. If $a+b=2$ and $a^2+b^2=3$, determine a^3+b^3. The simpler the algebra, the better the solution.

2. Consider a regular pyramid with a square base which has its top cut off by a plane parallel to the base. Such a figure is called a **frustum**. See the figure below. Suppose the square base has side length 2, and each of the slanted sides like AP has length 1. Finally, suppose the angle between AP and the plane of the base is $45°$.



a) Find the volume of the frustum.

b) Find the surface area of the frustum.

c) Determine the angles of the face $ABQP$.

d) Now suppose instead that the the angle between AP and the plane of the base is $60°$. Find the area of face $ABQP$.

3. Consider any sequence of 0's and 1's, for instance the length 8 sequence

$$1,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0.$$

A **block** is a consecutive sequence of the same digit that is not part of a longer such sequence. For instance, in the sequence above, the 1's appear is blocks of length 2 and 1, and the 0's appear in blocks of length 3 and 2.

a) Determine the number of sequences of 0's and 1's of length 10 in which every block of 1's has even length.

b) Determine the number of sequences of 0's and 1's of length 10 in which every block of 1's has odd length.

In both parts, brute force counting by hand or by computer is not allowed. The more direct and elegant your solution, the more points you get. Also, for each part you have to decide whether to include the all zeros sequence of length 10 in your count. Whatever you decide, explain your reasoning briefly.

4. Let E^+ be the set of even positive integers. A number is **prime within** E^+ if it cannot be factored within E^+. For instance, 6 is prime within E^+; sure, $6 = 2 \cdot 3$, but 3 is not in E^+. On the other hand, 4 is not prime within E^+ because $4 = 2 \cdot 2$.

a) Prove that every number in E^+ has a prime factorization within E^+. That is, prove that every number in E^+ is either prime within E^+ or factors into two or more numbers that are prime within E^+.

b) Find a number in E^+ that has 2 different prime factorizations within E^+ (different primes, not just a different order). Why is it interesting that you can find such a number?

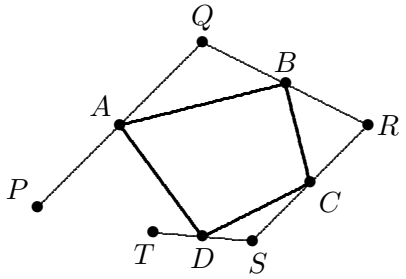
c) Determine with proof the set of all numbers in E^+ that have 2 different prime factorizations within E^+.

5. In this problem, $ABCD$ is an arbitrary but fixed quadrilateral in the plane. The point P can change.

Pick some point P anywhere on the plane. Then Q, R, S, T are defined as follows; see the figure below.

- Q is the point such that A is the midpoint of segment PQ;
- R is the point such that B is the midpoint of segment QR;
- S is the point such that C is the midpoint of segment RS;

- T is the point such that D is the midpoint of segment ST.



a) Draw this figure for various starting points P (but the same points A, B, C, D). Accuracy will help.

b) Make and prove a conjecture about the segment PT.

c) Under what circumstances is $P = T$?

6. You have a spinner that lands on 1, 2 or 3 with equal probability. You are allowed to spin as many times as you want, summing your scores as you go, *except* if you ever land on 1, you lose all your points (your score is now 0) and you can't spin anymore. If you don't get a 1, you can stop spinning whenever you want.

a) What is the probability that you can achieve a score of exactly 8?

b) What is the probability that you can achieve a score of at least 8?

c) Now you use this spinner to play the following game with a friend. You alone spin until you get wiped out with a 1 or until you choose to stop. Then your friend gets to spin (with the same wipeout rules) and tries to beat or tie your score. You want to maximize the probability that you end up with a higher score than your friend. What is that probability, and what is your strategy to achieve it?

d) (Optional) Think up some other variation of this game (either a solitaire version or 2-person version) and analyze it.