MathPath 2012 Qualifying Test

Instructions. Submit all those parts of the 7 problems below where you feel you have made good progress. If you make good progress on at least 3 problems, you should definitely apply, although more may be needed for admission. We do not look favorably on very good solutions to a few problems and nothing submitted for others, because we want to see your thinking on a variety of problems.

You may need to think about some problems for many days. You may ask other people to help you understand the statements of the problems, but the actual solutions must be your own. You may not work with others, including online acquaintances and friends who may also be applying. *If you have seen any of these problems before, still submit them, but let us know.*

Write your solutions on standard American letter paper (8.5×11) or the similar A4 international paper, ruled or unruled. Use only one side of each sheet, and write your name at the top of each. Your work must be dark enough and clear enough to xerox. Please start each problem on a new sheet. The sheets and the problems should be numbered. You need not copy the statements of problems. Your solutions should show all the steps in your reasoning and in your computations. The steps are more important than the answer.

You may also submit your work by email, as scans of handwritten work or as pdfs of documents created electronically (say with TEX or LaTEX or the math facility within MS Word). In any event, you must still number pages, start each problem on a new page and submit a file dark enough to xerox. Do not create an electronic document if this means you will skimp on algebra or on figures – tempting since complete work is harder to create electronically.

Communication is an important part of MathPath. Imagine you are writing to someone far away who knows about as much mathematics as you but needs more explanation. You want to make your written solutions appealing and easy to read and understand; this person can't talk to you for clarification. In particular, long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred (but not if they are shorter simply by leaving out reasons). So, if your first solution to a problem is long and complicated, take time to see if you can find a short direct solution, and submit the shorter solution only. Mathematicians say that such short direct solutions are *elegant*.

Generally we think that earlier problems are easier than later problems, but in all problems the last part is generally much harder. In any event, difficulty is in the eye of the beholder. Also, all problems can be done with paper and pencil.

Some submitted solutions will be displayed and discussed at camp (with names removed) as examples of good and not-so-good mathematical writing.

Where to submit. At most times send your hardcopy solutions to

MathPath
c/o Prof Maurer, Math/Stat
Swarthmore College
500 College Ave
Swarthmore PA 19081-1390 (USA)

However, during January–March 2012, Prof Maurer will be teaching in Germany. Although Qualifying Test submissions will be forwarded to him every few weeks, it is better during this period if you can submit electronically, as that always reaches him directly. To submit electronically, send email to smaurer1@swarthmore.edu, where "1" is the digit one. Include your work as an attachment. For the requirements for acceptance of scanned tests, click here.

However you submit your work, always keep a copy.

Scoring. You will not be told your score, because this is not a contest with a winner but instead a qualifying test where scores are used only to help determine which applicants have the mathematical qualities needed to enjoy and succeed at camp. Each problem is worth at least 10 points, with more difficult parts given more points.

Also, a correct solution need not result in a perfect score. For instance, a correct solution to a 10-point problem, by brute force with weak writing, might get only 5 points. An elegant, well written solution would get 10, or even more if interesting generalizations are proved.

For more discussion and examples of good and not-so-good solutions from earlier MathPath tests, click HERE. In response to questions we receive about this test, we may post some clarifications. Look for them HERE.

1. For positive integers m and n define $m \uparrow n$ as follows:

$$m \uparrow 1 = m$$
$$m \uparrow (n{+}1) = m^{m \uparrow n}.$$

For instance, $3 \uparrow 2 = 3^{3\uparrow 1} = 3^3 = 27$.

a) Evaluate $2 \uparrow 4$.

b) Find, with justification, the smallest integer n so that $2 \uparrow n$ is greater than a googol (1 followed by 100 zeros).

c) A googolplex is 10^{googol} (10 raised to the googol power). Find, with justification, the smallest integer n so that $2 \uparrow n$ is greater than $M = \text{googolplex}^{\text{googolplex}}$.

2. Consider sequences made by using one or more of the integers from 1 to n.

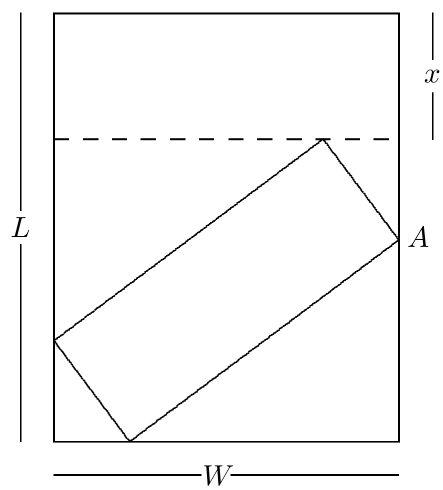
a) List all the strictly increasing sequences when $n = 4$. For instance, 1 by itself is a strictly increasing sequence (a sequence of length 1), as is $2, 4$, but $3, 2, 4$ and $1, 1, 1, 2$ are not, because neither is strictly increasing (each number has to be bigger than the previous one).

b) List all the strictly increasing sequences which alternate from odd to even when $n = 5$. The first term can be either odd or even. So for instance, 3 by itself is such a sequence, as is 1,2,3,4,5, but 1,3,4 is not because 1 and 3 are consecutive and both odd.

c) Now let n be an arbitrary positive integer. How many strictly increasing sequences are there using one or more of the integers from 1 to n?

d) Same question as c), but now the sequences have to alternate odd even as well (again starting with either odd or even).

3. A friend says she can take a rectangular sheet of paper and cut off a strip along the top of just such a size that it fits perfectly at an angle in the remaining sheet. See the figure, where the strip is the section above the dotted line.



a) Determine if your friend is right for American letter paper, with $L = 11$ and $W = 8.5$. If she is right, determine the size or sizes of strips that work. (They all are 8.5 long; what is the other dimension, shown as x in the figure?)

Note: The figure does not prove that there is a solution, because it is not quite drawn to scale, as you can check with a ruler.

b) Is your friend always right?

c) If this problem has a solution, the point A in the figure has an interesting property. Discover the property and prove it geometrically.

4. A function is **even** if its graph is symmetric around the y-axis, that is, whenever (a, b) is on the graph, so is $(-a, b)$. For instance, $y = x^2$ is even. A function is **odd** if its graph is symmetric around the origin, that is, whenever (a, b) is on the graph, so is $(-a, -b)$. For instance, $y = x^3$ is odd.

a) Show that the function $f(x) = x^2 - 3x + 4$ can be written as the sum of an odd function and an even function. Find specific odd and even functions that add to $f(x)$. If you find more than one pair that add to $f(x)$, show all pairs you find.

b) Find a specific odd and even function that add to $g(x) = 1/(x^2 + 2x + 2)$.

c) Is there a general theorem here? State and prove as general a result as you can.

5. Consider the spiral array of the positive integers below. If you look at the integers in order $1, 2, 3, \ldots$ you will see that they spiral out in increasing square rings. The spiral can be continued forever.

101	100	99	98	97	96	95	94	93	92	91	
102	65	64	63	62	61	60	59	58	57	90	
103	66	37	36	35	34	33	32	31	56	89	
104	67	38	17	16	15	14	13	30	55	88	
105	68	39	18	5	4	3	12	29	54	87	
106	69	40	19	6	1	2	11	28	53	86	
107	70	41	20	7	8	9	10	27	52	85	
108	71	42	21	22	23	24	25	26	51	84	
109	72	43	44	45	46	47	48	49	50	83	⋮
110	73	74	75	76	77	78	79	80	81	82	123
111	112	113	114	115	116	117	118	119	120	121	122

Imagine the usual cartesian coordinates placed on the plane, with point $(0, 0)$ in the middle of the cell containing 1, and point $(1, 0)$ in the middle of the cell containing 2. We say that 1 is in the $(0, 0)$ cell and 2 is in the $(1, 0)$ cell. Likewise, 10 is in the $(2, -1)$ cell.

a) What integer is in the $(-2, 8)$ cell? Justify your answer.

b) Consider the horizontal half line starting with 24 in cell $(1, -2)$ and going right. None of the numbers shown (24, 25, 26, 51, 84) are primes. Prove that this is true forever: none of the integers on this half line are primes.

c) Circle all the primes in the array. (A bigger copy of the array appears on a separate page, on which you can show your work.). Describe any patterns you find.

d) There are a lot of patterns in this array (not just about primes) and some of them continue forever. Explore! Submitting at most 2 pages, state the most interesting patterns you discover, and show proofs when you have them.

6. Let p, q be twin primes, that is, they differ by 2. Prove: $p^p + q^q$ is divisible by $p + q$.

7. Construct a set of positive integers by first including the numbers 2, 3 and 4. Then successively check each positive integer beyond 4 and include it in the set if it is equal to the sum of two distinct numbers already in the set *and* it is also equal to the product of two distinct numbers already in the set. Thus 5 would not be included in the set, but 6 would, because $6 = 2 + 4$ and $6 = 2 * 3$.

a) Prove that the set contains every power of 2.

b) Give a complete description of the set; come up with a condition such that a number is in the set if and only if it is of that form. Prove that you are right.

Note: This problem was created by a student in the MathPath Problem Writing course in 2010.

— end of problems —

101	100	99	98	97	96	95	94	93	92	91	
102	65	64	63	62	61	60	59	58	57	90	
103	66	37	36	35	34	33	32	31	56	89	
104	67	38	17	16	15	14	13	30	55	88	
105	68	39	18	5	4	3	12	29	54	87	
106	69	40	19	6	1	2	11	28	53	86	
107	70	41	20	7	8	9	10	27	52	85	
108	71	42	21	22	23	24	25	26	51	84	
109	72	43	44	45	46	47	48	49	50	83	⋮
110	73	74	75	76	77	78	79	80	81	82	123
111	112	113	114	115	116	117	118	119	120	121	122