MathPath 2008 Qualifying Test

Instructions. Do as many of the problems below as you can. If you can make good progress on at least 3 problems, you should definitely apply, although more may be needed for admission. You may need to think about some problems for many days. You may ask other people to help you understand the statements of the problems, but the actual solutions must be your own. You may not work with others, including friends who may also be applying and online acquaintances. (If you have seen any of these problems before, still submit them, but let us know.)

Use standard letter paper (8.5×11) or the similar A4 international paper, ruled or unruled. Your work must be dark enough and clear enough to xerox. Write on one side only. Please start each problem on a new sheet. The sheets and the problems should be numbered. You need not copy the statements of problems. However, your solutions should show all the steps in your reasoning and in your computations. The steps are more important than the answer. Correct answers without supporting reasoning will receive no credit.

Communication is an important part of MathPath. Imagine you are writing to someone far away who knows about as much mathematics as you but who has not thought about these problems before. You want to make your written solutions appealing and easy to read and understand; this person can't talk to you for clarification. In particular, long solutions with lots of cases are hard to follow. Shorter, more direct solutions are preferred (but not if they are shorter simply by leaving out reasons). So, if your first solution to a problem is long and complicated, see if you can find a short direct solution, and submit only that. Mathematicians say that such short direct solutions are *elegant*. Also, generalizations and additional elegant solutions are welcome and may count significantly.

Some submitted solutions will be displayed and discussed at camp (with names removed) as examples of good and not-so-good mathematical writing.

For more discussion and examples of good and not-so-good solutions from earlier MathPath tests, click HERE. For any posted hints and clarifications, click HERE.

1. Take the first four powers of 2, namely 1, 2, 4, and 8, and form all possible expressions that can be obtained by adding or subtracting all four of these numbers together. For example,

$$+1+2-4+8 \qquad \text{and} \qquad -1-2-4-8$$

are two such expressions, which evaluate to 7 and -15, respectively.

a) How many different values can be obtained in this manner, and what are they?

b) Based on your previous answer, make a conjecture as to what will happen if we use the first $n+1$ powers of 2 instead, that is, $1, 2, 4, \ldots, 2^n$. If you are able, prove your conjecture.

2. A teacher told a student: "Our courtyard has more than one tree, and each tree contains more than one bird. Furthermore, each tree has the same number of birds." The teacher then said how many birds total were in the courtyard. Based on this information the student was able to determine without any doubt the number of trees. Given that the total number of birds was between 200 and 300, how many trees were in the courtyard?

3. In a certain town, each pair of people are friends or not. Prove or disprove: Some two people in the town have the same number of friends in town.

4. A parallelogram has sides of length 6 and 10 and one angle is 60°. The four perpendicular bisectors of the sides of this parallelogram form another parallelogram. Determine its perimeter. (This problem was created by a student at MathPath 07, in a problem writing seminar.)

5. A polynomial $p(x)$ has the properties that

- When it is divided by $x^2 - 3x + 2$, the remainder is $2x - 1$;
- When it is divided by $x^2 - 5x + 6$, the remainder is $3x - 3$.

Find the unique polynomial of lowest degree that meets these conditions. Why is it unique?

6. (double credit) There is a curious formula for the volume of solids. First, rest the base of the solid on a horizontal plane, then rest a second horizontal plane on top of the solid, and then slice the solid with another horizontal plane halfway between the other two. See the figure below, where the solid is a cone. Let A_b, A_m, A_t be the areas respectively of the cross sections of the solid created by the bottom plane, the middle plane, and the top plane. (For instance, in this cone $A_t = 0$, since the top plane intersects the cone only in a point.) Then the formula claims

$$V = \frac{h}{6}(A_b + 4A_m + A_t). \qquad (1)$$

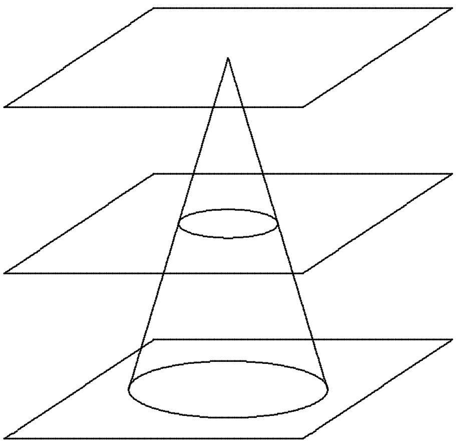
where h is the height of the solid.

a) Verify that this formula is correct for cones. That is, verify that (1) simplifies in this case to the usual formula for the volume of a cone.

b) Verify this formula for spheres. (For spheres, the bottom plane also intersects the solid in a point.)

c) Verify this formula for a truncated square pyramid, that is, a pyramid with square base where some amount of the top has been cut off parallel to the base and discarded.

d) Can you find any solids for which formula (1) fails? Show your reasoning or computations for all solids you consider.



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