

MathPath Breakout Catalog 2018

v.3, 6/21/18

Week 1, Morning

Intro to Contest Problem Solving, Steven (Steven Alexander)

We will delve into mathematics and approaches for solving problems at about the level of AMC 8-AMC 10. Each day, we will focus on a topic among number theory, algebra including powers, geometry and counting, with extra challenges from previous days. Daily practice will include students sharing techniques. *1-2 stars*

Elementary Graph Theory, Dr B (Jane Butterfield)

Most people think of a "graph" as a visual representation of data - a function, assorted information, etc. In the study of graph theory, we define a "graph" to be a set of points, called "vertices", and a set of lines connecting two of those "vertices", called "edges". Graphs of this sort can be used to diagram, understand, and solve many mathematical problems; some of which may surprise you! In this breakout, we will be investigating the fundamentals of graph theory, and discovering problems that can be solved by being diagrammed as graphs. *1 star*

Mathematica I, Dr C (David Clark)

Mathematica is a popular computer software program used by mathematicians and scientists. We will learn how to use Mathematica to perform basic mathematical operations, solve equations, create lists, and generate 2D graphics. No computer programming experience is necessary. (At the end of camp, each MathPath participant will receive a free copy of Mathematica.) *1 star*

Induction, Prof D (Matt DeLong)

If you line up infinitely many dominoes on their ends, with each one close enough to the previous one, and knock over the first, then all infinitely many fall down. This is the essence of mathematical induction, the main proof technique when you have infinitely many statements to prove indexed by the integers, such as "1-ring Towers of Hanoi can be won. 2-ring Towers of Hanoi can be won.... 487-ring Towers of Hanoi can be won...."

Every budding mathematician needs to know mathematical induction, and it's a great proof technique to learn first, because it has a standard template (unlike most proof techniques) and yet leaves room for an infinite amount of variety and ingenuity. Thus this course is one of MathPath's foundation courses.

But don't take my word for it. Here is what a MathPath student wrote on an AoPS MathPath forum:

I took induction last year, and I knew induction before the class. But it was very well taught, and I learned how to write proofs by induction, which was very valuable on the USAJMO. I'd recommend this class to anyone who wants to learn about induction, whether you know it at the beginning or not.

The point: Even if you have done induction before, you don't really *know* induction, because it can be used in so many ways in so many parts of mathematics. Every mathematician will probably do 1000 inductions in his/her life. Get 50 under your belt in this course. *2 stars*

Number Theory I, Joyce (Joyce O'Halloran)

Number theory has delighted young mathematicians throughout the years due to the accessibility of its ideas and the ingenuity of its techniques. This introductory level breakout presents the foundations of the subject, starting with divisibility and moving on to the Euclidean Algorithm, primes, factoring, counting divisors, perfect numbers, and relatively prime integers. We will conclude with a first look at congruences and modular arithmetic. *1 star*

Analytical Geometry, Dr T (George Thomas)

Francois Vieta's introduction of literal symbols for representing a class of numerical values was followed in 1637 by Rene Descartes' application of it in Geometry to represent a point and for the representation of lines and curves by equations. He set up an oblique coordinate system and then the rectangular coordinate system. A first degree equation in x and y gives a straight line in this system, and conversely, a straight line has a first degree equation. The discussion goes very briefly over the various forms of the equation of a straight line, and an example of how algebra is used to "analyze" geometric construction problems. The area of a triangle is obtained in two ways: In terms of the coordinates of the vertices and in terms of the equations of the three lines constituting the sides. Equations of the various conics are obtained from the Homogeneous General Equation of the Second Degree. The method of transformation of coordinates to convert a second degree equation to its most simplified form is illustrated. (IF TIME PERMITS: The simplified form is related – "affinis" – to the original and this leads in to a discussion of affine transformations and affine geometry. Whereas an ellipse is transformed in this manner to another ellipse and not, for instance, a hyperbola, there is a transformation, more general, that takes any conic to any other conic. It is the "projective" transformation – all conics are circles when viewed from the vertex of an infinite cone. This example is used to point out the role played by Analytic Geometry in the mathematical theory of perspective for curves: when a curve is specified an equation, the equation of the perspective view is obtainable by suitably transforming x and y .)

We show that given any five points in the plane there is a unique conic passing through them. Equations of tangents, normals, subtangents, and subnormals are obtained for the circle first and then for the other conics. Poles and Polars, and the "correlation" between pole and its polar for a conic are shown. *Pre-requisites: Algebra 2 and Trigonometry. 3 stars*

Week 1, Afternoon

Basic Counting (Combinatorics), Dr B (Jane Butterfield)

For students who know a little about how to count, but want to know more and get better. We start with the basics: sum rule, product rule, permutations, combinations, the binomial theorem. Then we learn about "combinatorial arguments", sometimes called proof by story. Time

permitting, we will look at inclusion-exclusion, and pigeon-hole arguments. This course is a good prerequisite for various later courses that involve counting. *2 stars*

Error-Correcting Codes, Dr C (David Clark)

If you've ever heard bad static in a phone call or seen strange blocks appear on your TV, then you needed an error-correcting code! A code is not a secret -- at least not the kind of code that we'll be playing with. (For that sort of coding, take the Cryptology course.) Our codes are exactly the opposite: A way of writing a message that makes it easier to understand and protects it from mistakes. In this breakout, we'll transmit messages to each other and take turns playing the part of static: the evil corrupter of messages. We'll learn about vectors, matrices, and how to use them to help cell phones, computers, and even deep-space probes to overcome mistakes and make your messages understood. *Number theory I recommended as a prerequisite but not required. 2 stars*

Elementary Logic, Prof D (Matt DeLong)

If you attend MathPath, you cannot be a cow. Bessie is a cow. Most of us would correctly deduce that Bessie cannot attend MathPath, but how do we know this, and how can we prove it? Logic serves as the foundation for our reasoning, setting out formally the rules we understand intuitively. Much as algebra allows us to generalize relationships among numbers by using variables to represent quantities, we will use symbols to represent propositions, demonstrating the form or structure of an argument (premises leading to a conclusion). Then we will formally prove these arguments using rules such as Modus Ponendo Ponens, Modus Tollendo Tollens, and Reductio Ad Absurdum. We will explore truth tables, semantic trees, symbolic logic proofs, and potentially predicate logic. Also, there are paradoxes which seem to defy common sense. We shall see how putting some paradoxical examples into the regimented forms demanded by logic causes the paradoxical element to disappear. *1-2 stars*

Hyperbolic Geometry, Dr T (George Thomas)

This is an ideal course for the future mathematician in that it combines history and the axiomatic method. The course begins with a discussion of Neutral Geometry – axioms, the Exterior Angle Theorem, Alternate Interior Angle theorem and the Saccheri-Legendre Theorem. Next: Euclid's Fifth postulate ("Fifth") is added; its equivalence with the Euclidean Parallel Postulate and the Converse of the Alternate Interior Angle Theorem is shown.

The efforts of two millennia to prove the Fifth from the Neutral Geometry postulates and the introduction of the hyperbolic postulate. H. Liebmann's proof: The area of a singly asymptotic triangle is finite. The finiteness of triangular area and Gauss's proof of the area of a triangle in terms of angular defect. The Bolya formula connecting the angle that a perpendicular of given length to a given line makes with a parallel to the given line is derived. Bolya's construction of ultraparallels. The Klein, Poincare and Beltrami models. *Pre-requisites: Trigonometry. 3 stars*

Heavenly Mathematics, Glen (Glen Van Brummelen)

How were the ancient astronomers able to find their way around the heavens without anything even as sophisticated as a telescope? With some clever observations and a little math, it's amazing how much you can infer. Following the footsteps of the ancient Greeks, we will eventually determine the distance from the Earth to the Moon...using only our brains and a meter stick. Along the way, we will develop the fundamentals of the subject the Greeks invented for this

purpose, now a part of the school mathematics curriculum: trigonometry. Scientific calculators are encouraged. *2 stars*

Taming the Torus, Dr. V (Sam Vandervelde)

Torus is to bagel as sphere is to cantaloupe. The torus presents a splendid variety of mathematical aspects for our examination. We will discover the geometric formulas associated with a torus, catalog the topological properties of a torus, investigate graph embeddings on a torus, even explore the possibilities for art and puzzles with a torus. *1 chili pepper*

Week 2, History Pullout for Returnees

Gaussian Integers and Dedekind's Ideal Numbers, Prof D (Matt DeLong)

During the nineteenth century, mathematics became more abstract, formal and rigorous. Within algebra and number theory, mathematicians shifted their focus from algorithms for solving various equations to the underlying structures of various number (and number-like) systems. Efforts to identify all integer solutions to certain number theoretic equations led to generalizations of the ancient and seemingly straightforward concept of an integer. A key player in this effort was Richard Dedekind, whose work led to the concepts of "ideals" and "rings" in abstract algebra. In this history pullout, we will follow along with Dedekind as he thinks deeply about the ordinary integers, and in particular the essential properties of their primes. We will compare to the Gaussian Integers ($\mathbb{Z}[i]$) and their corresponding Gaussian Primes, also exploring their connections to certain results in number theory. Finally, we will consider Dedekind's treatment of indecomposables in $\mathbb{Z}[\sqrt{-5}]$, his creation of an "ideal," and the mathematical after-effects.

Week 2, Morning

Graph Ramsey Games, Dr B (Jane Butterfield)

It would be surprising if there were ten students in my class who had the same birthday. It would not be surprising if there were ten students in Massachusetts who had the same birthday. That is because when things are large we expect patterns to show up. Ramsey Theory answers the question "how big does the group have to be for a particular pattern to always happen?" In the context of graph theory, we might ask how many edges a graph can have before it has to contain a clique of size three. Questions like this form the basis of Graph Ramsey theory. In this breakout session, we will answer some Graph Ramsey questions, and learn just how hard the unanswered ones are. Then we will spend the rest of our time playing games -- I mean, "considering Graph Ramsey Theory from a competitive perspective." Some of these games will be between two intelligent players, and some will be between an intelligent player and a random adversary. We will show how these games are related, and try to decide who wins them. This area is still very recent, and many easy-to-ask questions are still unanswered (and are possibly very difficult).
Prerequisite: It would be nice if you know some graph theory but not necessary. 2-3 stars

Mathematica II, Dr C (David Clark)

This course will expand on the topics covered in Mathematica I. We will learn how to define our own functions, how to create interactive applets using the Manipulate command, and how to generate 3D graphics. You will have the opportunity to propose a project of your own design and create it in Mathematica. Exercises will be more challenging than the ones in Mathematica I.

Prerequisite: Mathematica I, or an interview with the instructor about your programming experience. 2-3 stars

Number Theory II, Prof D (Matt DeLong)

Suppose a fruit stand owner wants to arrange oranges neatly. If the oranges are arranged in rows of 5 then there are 2 left over, if arranged in rows of 6 then there is 1 left over, and if arranged in rows of 7 there are 3 left over. How many oranges could the owner have? Is there a unique solution?

Answering this question requires a deeper understanding of modular arithmetic than we achieve in NT1. So in NT2 we discuss inverses mod m , the Chinese Remainder Theorem, Fermat's Little Theorem, Wilson's Theorem, the Euler phi-function and Euler's extension of Fermat's Little Theorem. These results will allow us to answer other questions as well, such as: How many ways can 1 be factored? What happens when one repeatedly multiplies a number by itself? And because the answers to such questions are so interesting, we can all enjoy quick divisibility tricks. *2 stars*

Projective Geometry, Dr T (George Thomas)

Euclidean Geometry is the geometry of lines and circles: its tools are the straight edge (unmarked ruler) and compass. There is a geometry where its constructions need only a straight edge. In this geometry a straight line joins two points and two lines never fail to meet. It turns out to be simpler than Euclid's but not too simple to be interesting. This is Projective Geometry which turns out to be the mother of all geometries, for Euclidean, Elliptic, Hyperbolic and Minkowskian Geometries are but special cases of it.

The course begins with Pappus's ancient theorem. The discussion then proceeds as follows: The new Geometry's axioms, the principle of duality, the concept of projectivity, Harmonic sets, The Fundamental Theorem of Projective Geometry, and Desargue's Theorem. *Pre-requisites: None. 3 stars*

Proof by Story, Dr V (Sam Vandervelde)

Armed with a bit of practice and a proper understanding of what certain well-known numbers mean, it becomes possible to prove intimidating identities involving these numbers simply by telling the right story! For instance, in this breakout we will discover that Fibonacci numbers have a combinatorial meaning: they count the number of ways to accomplish a certain task. This view of a Fibonacci number, which goes well beyond the basic algorithm for obtaining a Fibonacci number as the sum of the two previous, enables us to establish a surprisingly large number of clever relationships among them. *2-3 chili peppers*

Continued Fractions, Dr Z (Paul Zeitz)

Base ten is of course arbitrary, stemming only from the fact that most humans have five fingers on each hand. But even binary is not “natural.” In this breakout, I will convince you, I hope, that the only truly natural representation of a real number is as a continued fraction: a possibly infinite composition of two simple operations (adding a positive integer and taking the reciprocal). For example, if we start with zero, and do "add one, then take the reciprocal" several times, we will get, in succession, $1, 1, 2, 1/2, 3/2, 2/3, 5/3, 3/5, 8/5, 5/8$, etc. Look familiar?

Playing around with continued fractions is fun, but why should you care? It turns out that continued fractions live in a rich mathematical crossroads, where number theory, non-euclidean geometry, linear algebra, approximation theory, and probability all visit. By exploring this topic, we will learn about some of these connections. That's what happens when you investigate the natural way to represent numbers! *2 stars*

Week 2, Afternoon

Eulerian Graphs, Dr B (Jane Butterfield)

What is the best route to take through my neighbourhood this Halloween? Can I walk through this building, locking each door behind me, without getting stuck? How can I search this game's level without missing any treasure chests? Can I embroider this pattern so that it looks good on both sides of the fabric? In this breakout session, we will learn how to model each of these questions with graph theory. Then we will learn -- and prove -- a powerful theorem credited to Euler, as well as some extensions that we will need in order to answer all of these (and more) questions! *Prerequisite: some previous experience with graph theory, such as from taking the Elementary Graph Theory Breakout. 1-2 stars*

Guessing Games, Dr C (David Clark)

I'm thinking of a secret whole number from 1 to 10. How many yes-or-no questions do you need to ask in order to figure it out? What if my number is 1 to 1,000,000? How about 1 to n -- and how do you know that you've asked the fewest questions possible? In this breakout, we will start with those problems and then go much farther. For example: What if I'm allowed to lie -- but only once? Twice? Prerequisites include comfort with simple proofs and Basic Counting (Combinatorics). Homework will regularly involve playing guessing games against a friend and implementing a strategy we've learned in class as you do so. Come prepared to play! *3 stars*

Elliptic Geometry, Dr T (George Thomas)

In this course we explain why there are exactly three 3-dimensional constant-curvature geometries – Euclidean, hyperbolic and elliptic. Whereas the Euclidean and hyperbolic geometries are neutral geometries, we show that Elliptic Geometry is not a Neutral Geometry. However, Elliptic Geometry complements the Euclidean and Hyperbolic geometries in terms of the number of parallels, namely none. We discuss Elliptic Geometry using two models – Spherical Geometry and the Klein Circle model – to show also that it is a projective space with an elliptic metric. We also prove some fundamental properties of a sphere including that a geodesic is a great circle on the sphere. *Pre-requisite: Trigonometry. Those who took Spherical Geometry*

will find some overlap due to the use of the sphere in building the model of Elliptic Geometry. 3 stars

Spherical Trigonometry, Glen (Glen Van Brummelen)

To find your way through the heavens, along the earth, or across the oceans, you need mathematics. But the math you learn in school mostly takes place on a flat surface --- not the sphere of the heavens, or the earth. To navigate properly, we develop a completely new trigonometry that allows us to find our way around a sphere. We shall discover surprising symmetries and a rich world of theorems, many of which are beautiful and unexpected extensions of some of the most familiar geometric theorems we have learned in school. Scientific calculators are encouraged. *3 stars*

Four Star Problem Solving, Dr V (Sam Vandervelde)

This breakout is designed for students with substantial contest problem solving background who have a solid working knowledge of algebra, geometry, counting, and number theory. The purpose is to gain experience with a variety of advanced tools and techniques within these areas that often surface in contests such as the AMC-10. We will examine each topic in turn, culminating in a class contest on the last day to apply what we've learned. *4 chili peppers*

If Gauss and Abel did it, you can, too!, Dr Z (Paul Zeitz)

You probably know that Karl Gauss constructed the regular 17-gon in 1796, resolving a question that had been open for about two thousand years. This construction required solving a 16th-degree polynomial equation using "only" repeated square roots. At the time of Gauss's great achievement, it was still unknown just which polynomial equations were "solvable by radicals:" the only known methods stopped at 4th-degree ("quartic") polynomials. But a few decades later, Niels Hendrik Abel showed that 5th-degree ("quintic") polynomials, in general, cannot be solved with radicals.

In this breakout, we will follow in the footsteps of these great thinkers. We will essentially follow Gauss's original methods, and learn to solve the "heptadecagon" equation. We will also investigate Abel's ideas, but using more modern methods. We will be scratching the surface of the fascinating branch of math known as Galois Theory. *Prerequisites: Knowledge of complex numbers and some number theory will be necessary. 3-4 stars*

Week 3, Morning

Intermediate Contest Problem Solving, Steven (Steven Alexander)

We will delve into mathematics and approaches for solving problems at about the level of the harder problems of AMC 10, while also discussing making sure to get the less hard ones. We will consider number theory, algebra including polynomials and functions, geometry including trigonometry, and counting, each day focusing on one, with challenges from previous days' topics. We will have daily practice including sharing techniques. *2 stars*

Advanced Graph Theory, Prof B (Owen Byer)

We will study a variety of topics in advanced graph theory. Ramsey theory and graph colorings (including chromatic polynomials) are fun topics and active research areas. We will also look at necessary and sufficient conditions for the existence of Hamiltonian paths and circuits. Other possible topics include Hall's Theorem (on graph matchings) and Cayley's formula (number of labeled trees on n vertices). *Pre-requisite: Students should have previous experience with graph theory, counting, and induction. 3 stars*

Knot Theory, Prof D (Matt DeLong)

Knots have fascinated farmers, sailors, artists, and myth-makers since prehistoric times, both for their practical usefulness and for their aesthetic symbolism. For over 200 years, mathematicians have intensely studied knots, both for their own intrinsic mathematical interest and for their potential application to chemistry, physics and biology. Today, Knot Theory is an active field of mathematical research with many important applications. It is visual, computational and hands-on, and there are many easily stated open problems. This class will give an introduction to the fundamental questions of Knot Theory. It will take students from simple activities with strings to open problems in the field. It will answer such questions as, "what is the difference between a knot in the mathematical sense and a knot in the everyday sense?," "how do I tell two knots apart?," "how can I tell whether a knot can be untangled?," and "how many different knots are there?" *2 stars*

Compositions, Partitions and Fibonacci Numbers, Dr. K (Kathryn Nyman)

How many ways can one you write 4 as a sum of positive integers? If order matters, so that we consider $3+1$ and $1+3$ to be different, then the answer (8) is an easy application of basic combinatorics. But if we only care about the (unordered) *set* of addends, the answer is 5. This simple question leads to a rich, elegant and surprisingly challenging mathematical area on the intersection of combinatorics and number theory. We will look at integer partitions from several vantage points and take detours to visit the Fibonacci numbers. Along the way we will encounter beautiful bijections, clever counting, and tricky tilings. For this breakout, some previous experience with elementary combinatorics will be helpful. *3 stars*

Cryptology, Asia (Asia Matthews)

Have you ever made up a code to send a secret code to a friend, or solved the cryptogram in the newspaper? It turns out there's an entire area of math dedicated to this: cryptology is the study of making and breaking codes. We'll use modular arithmetic to cover classical codes, including shift ciphers, Vigenere ciphers, and substitution ciphers in general. By the end of the week, we'll also cover modern cryptosystems, which are used to keep our information secure online.

Prerequisite: Number Theory I. 2 stars

Euclidean Motions, Dr Y (Phillip Yasskin)

A Euclidean motion is a transformation of a 2-dimensional plane (or 3-dimensional space) which preserves lengths and angles. The well-known motions are translations, rotations and reflections. A less common motion is a glide-reflection. We will see that the set of Euclidean motions form a group and that any Euclidean motion can be written as a composition of translations, rotations and reflections or more simply as a composition of just reflections. We will also see how Euclidean motions can be written in terms of matrices. *2 stars*

Week 3, Afternoon

Inversions, Prof B (Owen Byer)

An inversion of the plane is a transformation of the points in the plane that preserves various properties of lines, circles, and the way they intersect each other. Inversions are a tool that can be used to provide elegant proofs of many geometric facts. They can be used to create interesting and artistic distortions of figures. Furthermore, properties of inversions can be combined with software like Geogebra to create dynamic figures (for example, Steiner's Porism), and we will explore these interactions. *Students taking this course should have some experience in proving theorems from geometry, though no specific knowledge is necessary. Geogebra experience would be helpful, though it is not required. 3 stars*

Discrete Morse Theory, Prof K (Prof Dominic Klyve)

Discrete Morse Theory is a new field of mathematics -- it was created just 20 years ago, and it is still rapidly developing. A special case of the more general theory involves graphs, or networks. We can think of a network as a map, and describe the height of each node and of all the lines connecting them. Discrete Morse Theory involves trying to understand and describe what the map looks like when we only look at the parts below a certain height. Students will learn the basics of this exciting new field of mathematics, and even have the chance to look at problems that nobody currently knows how to solve. *2-3 stars*

Complex numbers, Asia (Asia Matthews)

Complex numbers are just (linear) combinations of real and imaginary numbers! First, I'll tell you a story about how imaginary numbers arose naturally from studying geometry... and then algebra... and then functions, and how they were completely disregarded because of how unnatural they are. Then, we will see how many years of playing around with polynomials and looking for solutions led to an interesting realization: if we represent these imaginary things by some symbol, we can keep doing algebra and everything works out nicely. A few people decided to embrace the nonsense and they invented some imaginary geometry, and thus were born complex numbers! It turns out that really ugly looking algebraic representations of real systems (e.g. electrical circuits) are easily solvable by passing into the imaginary realm, doing calculations there, and then using only the real results. In this course we will play around with complex numbers. We will do some algebra with complex numbers, and see how they are related to vectors, to circles and triangles, to the number e , and to the sine and cosine functions. Some experience with polynomials will be helpful, but not necessary. It may sound really difficult, but it's just complex. 😊 *1-2 stars*

Cutting Cakes, and other Fair Division Questions, Dr K (Kathryn Nyman)

Whether dealing with property, inheritance, revenue, or taxes, the question of how to divide "goods" (or "bads") fairly among a group of people is a common problem. One delicious way to model "goods" is as a cake. At first glance, it seems easy to divide a cake fairly among n people: just cut it into n equal-sized pieces. But, while this might work for a homogeneous chocolate cake (without frosting??), consider a cake that is half chocolate and half vanilla, with sprinkles in one corner; that is, a cake where each person may value different parts of the cake differently. Now making sure each person gets a piece they are happy with becomes a curiously complicated problem. We will talk about what it means to be fair, and how mathematical algorithms can help

ensure an equitable division of tasty treats. There are no prerequisites except the ability to reason carefully and think logically about algorithms (sequence of steps). *2 stars*

Subcollection Sum Divisibility Theorems, Dr Y (Phillip Yasskin)

The goal of this session is to determine the values of k and n for which the following statement is true or false:

$P(k, n)$: For every collection, S , of n integers there is a subcollection, T , of k integers whose sum is divisible by k .

We will first look at examples where $P(k, n)$ is true. Next, we will determine all values of k and n for which $P(k, n)$ is false by giving counterexamples. And third we will determine all values of k and n for which $P(k, n)$ is true. *4 stars*

Three Star Problem Solving, Dr Z (Paul Zeitz)

We will work with a variety of "contest math" topics, with an emphasis on slowness. In other words, we won't focus on short-time-limit MATHCOUNTS questions, but instead work with questions that require logical arguments for their resolution. Our week will culminate in a "Math Wrangle" team contest, a hybrid of a debate contest and a wrestling match (minus the violence). *3 stars*

Week 4, Morning

Polygonal Areas and the Corset Theorem, Prof B (Owen Byer)

We all know the simple formula for finding the area of a triangle, but there are many other more interesting formulas, some of which (Shoelace and Corset Theorem) extend to finding the areas of quadrilaterals and general polygons. We will learn lots of geometry while exploring these formulas, including the fact that if every other vertex of a polygon is translated by a fixed vector, the areas of the original and resulting polygon are equal. *2-3 stars*

Cryptology, Dr. Lynn (Melissa Lynn)

See description in Week 3 morning. *Prerequisite: Number Theory I. 2 stars*

Sequences and Generating Functions, Asia (Asia Matthews)

A sequence is a list of numbers, and the sequences that we are interested in are those which have some pattern. First we will develop a few sequences of numbers from the sequence 1, 2, 3, 4, 5, 6, ... Then we will discuss how to describe a sequence, both implicitly and explicitly. For example, each term in the sequence above can be described as one more than the term before it (implicit), but we can also say that the 100th term of the sequence is the number 100 (explicit). Can you do find an implicit or explicit expression for the terms in the sequence 1, 5, 14, 30, 55, 91, ...? Sometimes doing this is easy, but more often it's difficult. Is it ever impossible? Sometimes finding an implicit expression is easier than finding an explicit one. (Consider, for example, the Fibonacci sequence.) Are there techniques that will always give us a solution?

Nope. But in this course we will learn a technique that works some of the time, and that's the best we can do so far. The technique involves infinite polynomials (generating functions), and we will have to do just a little bit of analysis, maybe some long division with polynomials, and definitely some partial fraction expansion. This course is heavy on the algebra, so come prepared to play around with abstract symbols. *3-4 stars*

The Essence of Calculus, Joyce (Joyce O'Halloran)

Is calculus about calculation? A typical calculus course would lead you to believe that. But they leave out the best stuff in this rich and beautiful area of mathematics! This is not a one-week calculus course; what you will experience in this session does not appear in calculus texts and we won't be deriving formulas or plowing through arduous symbol manipulations. What we will be doing is exploring the essence of calculus in the world around us (motion, topographical maps, etc.), using only basic geometry concepts. *1 star*

Isometries, Prof Rogness (Jon Rogness)

You've probably heard of rotations and reflections, which are special examples of what we call isometries. These functions always preserve lengths and angles; for example, if you draw a triangle and reflect it across a line, the result will be a new triangle which is congruent to the original triangle. You might know that there are more isometries than just reflections and rotations — in fact, in two dimensions there are exactly four types. But how can we prove that fact? How do we know our list is complete? In this breakout we'll answer those questions and more. Prerequisites: we'll make heavy use of matrices, vectors, and you should be very comfortable with matrix multiplication. You'll also need to know some basic unit-circle trigonometry, and be familiar with analytic geometry. You don't necessarily need to have taken the Analytic Geometry course at MathPath, but should be comfortable working with coordinates, points, and graphs of equations. *3 stars*

Polygon Differencing Games, Dr Y (Phillip Yasskin)

Put non-negative integers on the corners of a polygon and their differences at the midpoints. Connect the midpoints and repeat the process until it terminates. This is a Polygon Differencing Game. What are the final configurations? The final state is very different for a triangle and a square. So the question becomes: For which N does the N -gon Differencing Game behave like a square and which like a triangle? *3 stars*

Week 4, Afternoon

The Binomial and Poisson Distributions, Prof B (Owen Byer)

The Poisson Distribution is used to find the probabilities of the number of occurrences of some event in a fixed interval of space or time. Typical questions include the following.

1. What is the probability that there are two typographical errors on a random page of a given novel?
2. What is the probability that three different calls are made to an Emergency Response Unit in a given hour?

We will derive the formula for the Poisson probability distribution from beginning principles, discovering the binomial distribution and the number e along the way. We will examine many applications to real world phenomena. *Students taking this course should have experience with basic probability and familiarity with combinations and permutations will be very helpful.* 2-3 stars

Symmetries of Polygons, Dr. Lynn (Melissa Lynn)

A square has a lot of symmetry. You can rotate the square 90 degrees, 180 degrees, or 270 degrees, and it looks like the same shape. You can also reflect the square over a vertical, horizontal, or diagonal line through its middle. We'll investigate the symmetries of various regular polygons and study their structure. These symmetries give an example of a fundamental object in advanced mathematics, called a group. We'll look at a few other examples of groups, and see how these relate to each other. 3 stars

Origami, Asia (Asia Matthews)

The mathematical worlds that we choose to create and explore sometimes lead to surprising results. For example, if our world is made up of only circles and lines, the Greeks showed that while we can draw and bisect any angle we want (is this even true?), it is NOT possible to trisect an arbitrary angle. But strangely, if we begin with a square piece of paper, it IS possible to construct a trisection of an arbitrary angle. What is going on here? What do we have with the square piece of paper that we do not have with lines and circles? In this breakout session, we will play with paper folding to investigate this problem and other geometric results, we will fold checkerboards, discuss the fold-and-cut problem (proved less than 20 years ago) and we will discuss the intersection of mathematics and art. Come ready to apply yourself to the logical deductive process. 2 stars

The Shape of Space, Prof Rogness (Jon Rogness)

This breakout will explore ideas in the area of math known as topology, where a donut (torus) and coffee cup are equivalent objects. It will provide a nice followup for any students who have done the previous breakouts this year which covered the torus, although those courses are not prerequisites for this one.

We'll start by learning a few basic rules in topology. Then we'll look at the torus, either as a review or brief introduction depending on whether students were in those previous breakouts. Then we'll look at how to construct other two dimensional surfaces, like the Mobius strip and Klein Bottle. As it turns out, in one of the triumphs of mathematics, we can describe how to build all of the so-called "closed surfaces" by sewing together spheres, Moebius strips and donuts. After we deal with surfaces, we'll have the necessary skills to move on to three dimensions. That means we can think about different possible shapes for the universe -- i.e. the Shape of Space -- and explain why the picture below with the dodecahedra might be a model of how our universe is put together.

There won't be much in the way of computations or algebra in this breakout, but you should like thinking about three dimensional shapes. (For example, if you like looking at nets and figuring out what the resulting shape is, we'll be doing more complicated versions of that process.) 2 stars

Hands on History, Dr. Shell (Amy Shell-Gellasch)

Like to learn by doing? Then this session plays right into your hands (pun intended). In this breakout session, we will explore mathematical concepts of a historical nature by doing them! The topics covered will roughly follow the topics discussed in the week 4 History plenary sessions. We will built a simple sextant to explore how it was used by 19th century surveyors in determining the length of the meter; create a code device like the ancient Greeks; find out why the ancient Egyptian surveyors were called “rope stretchers”; and discover interesting facts about surfaces with Möbius. *1 star*

Special Relativity, Dr Y (Phillip Yasskin)

The Principle of Relativity is the understanding the laws of Physics need to give the same predictions regardless of the reference frame of the observer. The Principle of Special Relativity is based on the single postulate that the speed of light is the same in every reference frame which was experimentally observed in the 1887 Michelson-Morley experiment. We will investigate the foundations of special relativity from a geometrical perspective and find that geometry of Minkowski space replaces Euclidean space and Lorentz transformations replace or supplement rotations. Then we will resolve some of the classical paradoxes. *3 stars*

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