Convergence of the secant method

The secant iteration uses a secant line approximation to the function $f$ to approximate its root. Let $p$ be such that $f(p) = 0$, and let $p_{k-1}$ and $p_k$ be two approximations to $p$. Let us use the abbreviation $f_k \equiv f(p_k)$ throughout. If we take as our next approximation to $p$ the root of the (secant) line passing through $(p_{k-1}, f_{k-1})$ and $(p_k, f_k)$, then we get the secant method:

$$p_{k+1} = p_k - \frac{p_k - p_{k-1}}{f_k - f_{k-1}} f_k$$

(You can also arrive at this method by starting with Newton’s method and then replacing $f'(p_k)$ by $\frac{f_k - f_{k-1}}{p_k - p_{k-1}}$.)

We are interested here in analyzing the speed of convergence of the secant method. To that end, define the error at the $k$th step to be $e_k = p_k - p$. We will use a Taylor approximation $f(p + e_k) = e_k f'(p) + e_k^2 f''(p)/2 + O(e_k^3)$ under the assumption that $f'(p) \neq 0$. Then

$$e_{k+1} = p_{k+1} - p = p_k - \frac{p_k - p_{k-1}}{f_k - f_{k-1}} f_k - p$$

$$= \frac{(p_{k-1} - p) f_k - (p_k - p) f_{k-1}}{f_k - f_{k-1}}$$

$$= \frac{e_{k-1} f_k - e_k f_{k-1}}{f_k - f_{k-1}}$$

$$= \frac{e_{k-1} f(p + e_k) - e_k f(p + e_{k-1})}{f(p + e_k) - f(p + e_{k-1})}$$

$$= \frac{e_{k-1} (e_k f'(p) + e_k^2 f''(p)/2 + O(e_k^3)) - e_k (e_{k-1} f'(p) + e_{k-1}^2 f''(p)/2 + O(e_{k-1}^3))}{e_k f'(p) + e_k^2 f''(p)/2 + O(e_k^3) - (e_{k-1} f'(p) + e_{k-1}^2 f''(p)/2 + O(e_{k-1}^3))}$$

$$= \frac{e_{k-1} e_k f''(p)(e_k - e_{k-1})/2 + O(e_{k-1}^4)}{(e_k - e_{k-1})(f'(p) + (e_k + e_{k-1}) f''(p)/2 + O(e_{k-1}^2))}$$

$$= \frac{e_{k-1} e_k f''(p)}{2 f'(p)} + O(e_{k-1}^3)$$

We want to find $\alpha$ such that $|e_{k+1}| = C |e_k|^\alpha$, so we drop the cubic term and solve

$$|e_{k-1} e_k f''(p)| = C |e_k|^\alpha,$$

giving $|e_{k+1}|^{\alpha - 1} = D |e_k|$, where $D = |f''(p)/(2C f'(p))|$. Then $|e_{k+1}|^{\alpha (\alpha - 1)} = D^\alpha |e_k|^\alpha$. This forces $D = D^\alpha$ and $\alpha (\alpha - 1) = 1$. The only positive solution to the equation in $\alpha$ is $\alpha = (1 + \sqrt{5})/2 \approx 1.618$ (the golden mean). Furthermore, the asymptotic error constant must be $C = |f''(p)/(2 f'(p))|^{\alpha - 1} \approx |f''(p)/2 f'(p)|^{1.618}$.